# Lecture 31

## Chapter 7 Properties of Expectation.

- this chapter focuses on developing and exploring further properties of expected value.

**Prop:** Suppose $X$ is a RV s.t. $P(a \leq X \leq b) = 1$.
Then $a \leq E[X] \leq b$.

**Pf (Continuous case):**

$$\int_a^b a f(x)dx \leq E[X] \leq \int_a^b b f(x)dx$$

x bounded above by b

$$\Updownarrow$$

x bounded below by a

$$a\int_a^b f(x)dx \leq E[X] \leq b\int_a^b f(x)dx$$

$$a \leq E[X] \leq b.$$

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## 7.2. Expectations of Sums of RV's

**Prop:** Suppose $X$ and $Y$ have:

a) joint prob mass function $p(x,y)$, then

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)\, p(x,y).$$

b) joint density $f(x,y)$, then

$$E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y).$$

See the text for ~~proof~~.

<u>Special case</u>: $g(X,Y) = aX + bY$

Then

$$E[aX+bY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax+by)\, f(x,y)\,dx\,dy.$$

$$= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\,dy\,dx + b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\,dx\,dy$$

$$= a\int_{-\infty}^{\infty} x f_X(x)\,dx + b\int_{-\infty}^{\infty} y f_Y(y)\,dy.$$

$= aE[X] + bE[Y]$, whenever $E[X]$ & $E[Y]$ are finite.

An easy induction argument gives more generally that

$$E\left[c + \sum_{i=1}^{n} a_i X_i\right] = c + \sum_{i=1}^{n} a_i E[X_i].$$

Example: (Sample mean)

Suppose $X_1, \dots, X_n$ be independent, identically distributed (iid) with mean $\mu$. Then $\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}$ is called the sample mean and $E[\overline{X}] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] = \frac{n\mu}{n} = \mu$.

Ex: Unions of events.

Let $A_1, A_2$ be events.

Define $X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$

Observe that

$$1-(1-X_1)(1-X_2) = 1-(1-X_1-X_2+X_1X_2)$$
$$= X_1+X_2-X_1X_2$$
$$= \begin{cases} 1 & \text{if } A_1 \cup A_2 \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

Now, $E[X_i] = P(A_i)$.

So $E\Big[1-(1-X_1)(1-X_2)\Big] = P(A_1 \cup A_2)$.

S $E[X_1+X_2-X_1X_2] = P(A_1 \cup A_2)$

$E[X_1]+E[X_2]-E[X_1X_2] = P(A_1 \cup A_2)$.

$$\Rightarrow \quad P(A_1)+P(A_2)-P(A_1A_2)=P(A_1\cup A_2).$$

Same argument shows inclusion-exclusion for events $A_1, \dots, A_n$.

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Rmk: In general, it is <u>not</u> true that

$$E\left[\sum_{i=1}^{\infty} X_i\right] = \sum_{i=1}^{\infty} E[X_i]. \quad (*)$$

why? $E\left[\sum_{i=1}^{\infty} X_i\right] = E\left[\lim_{n\to\infty} \sum_{i=1}^{n} X_i\right]$

but the RHS is not necessarily

$\lim_{n\to\infty} E\left[\sum_{i=1}^{n} X_i\right]$ i.e. can't swap "E" and "lim".

Two cases where $(*)$ <u>IS</u> true

1) $X_i$ are all non-negative, i.e. $P(X_i \geq 0) = 1$.
2) $\sum_{i=1}^{\infty} E[|X_i|] < \infty$.

**Proposition:** Suppose that $X$ and $Y$ are independent. Then, for any functions $g(x)$, $h(y)$,

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

**Pf:** (Continuous Case). Let $f(x,y)$ be the joint density.

$$E[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)\,dx\,dy.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\, f_X(x)\, f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} g(x)f_X(x)\,dx \int_{-\infty}^{\infty} h(y)f_Y(y)\,dy$$

$$= E[g(X)]E[h(Y)].$$

**Defn:** For random variables $X$ and $Y$, we define the covariance of $X$ and $Y$, as

$$\mathrm{cov}(X,Y) = E[(X-E[X])(Y-E[Y])].$$

Note that

$$\mathrm{cov}(X,Y) = E[XY - Y\,E[X] - X\,E[Y] + E[X]E[Y]]$$

$$= E[XY] - E[X]E[Y]$$

It follows that if $X$ and $Y$ are independent,

then $\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y]$

$= E[X]E[Y] - E[X]E[Y] = 0.$

The converse is not true:

Ex: Let $X$ be uniform on $(-1,1)$. Let $Y = X^2$.

Then $\operatorname{Cov}(X,Y) = E[XY] - E[X]E[Y]$

$= E[X^3] - 0 \cdot E[Y].$

$= E[X^3] = \int_{-1}^{1} x^3 \cdot 1\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4}$

$= 0.$

So $\operatorname{cov}(X,Y) = 0$, but $X$ and $Y$ are clearly

not independent.